LIFTING LINE THEORY
AERO 2258A LLT Tutorial Example
Author: Hadi Winarto ; date: 10 May 2004 corrected 19 May 2004

This theory was developed basically to answer the following important aerodynamic question : Given the shape of a wing, how can we predict the aerodynamic properties of the wing? What is the most efficient wing shape for a given task such that the induced drag can be minimized?

The theory was developed by Prandtl in Germany and Lanchester in England at the beginning of the twentieth century as described below on the following web site
http://www.centennialofflight.gov/essay/Theories_of_Flight/Prandtl/TH10.htm

"***His 1904 paper raised Prandtl's prestige as an aerodynamicist. He became director of the Institute for Technical Physics at the University of Göttingen later in the year, where he worked with many outstanding students, creating the greatest aerodynamics research center of his time.***

In the years that followed, Prandtl began work on calculating the effect of induced drag on lift. Induced drag is the drag created by the vortices that trail an aircraft from the tips of its wings. These vortices, or whirling motions of fluid, affect the pressure distribution over the wings and result in a force in the direction of drag. Hence, induced drag is a kind of pressure drag. He worked with Albert Betz and Max Munk for almost ten years to solve this problem. The result was his lifting line theory, which was published in 1918-1919. It enabled accurate calculations of induced drag and its effect on lift.

In England, Prandtl's lifting line theory is referred to as the Lanchester-Prandtl theory. This is because the English scientist Frederick Lanchester published the foundation for Prandtl's theory years earlier. In his 1907 book Aerodynamics, Lanchester had described his model for the vortices that occur behind wings during flight. Prandtl's model for his theory was similar to Lanchester's, although Prandtl claimed that he had not considered Lanchester's model when he had begun his work in 1911.

During World War I, Prandtl created his thin-airfoil theory that enabled the calculation of lift for thin, cambered airfoils. It is still used today. He later contributed to the Prandtl-Glauert rule for subsonic airflow that describes the compressibility effects of air at high speeds. Prandtl also made important advances in developing theories of supersonic flow and turbulence."

The theory is based on the assumption that even though the flow around an aircraft's wing is really 3-dimensional, it may be satisfactorily approximated by a linear summation of flows around the elemental aerofoils, which makes up the overall wing, where the flow around each aerofoil is assumed to be 2-dimensional. This approach gives a reasonable result provided that the model flow takes into account the effect of the vortex sheet, which is shed at the trailing edge of the wing. The trailing vortex sheet induces a downwash velocity, which varies along the span wise direction.

The wing is assumed to be a flat plate lying on the x-y plane. Therefore, the theory does not take into account the wing's thickness distribution. It is also unable to handle any dihedral or sweepback angle. However, it is capable of modelling a tapered wing with geometrical and aerodynamic twists.

The problem needs the following input data:
1.Wing span, b.
2.Spanwise distribution of the following quantities
2.1 Sectional profile or aerofoil's chord length, c(y)
2.2 Aerofoil's geometric angle of incidence, α_G (y)
2.3 Aerofoil's zero lift angle of incidence, α_0 (y)
2.4 Aerofoil's lift curve slope, a (y)

Given the above data the theory must find the solution in the form of span wise wing load or lift per unit span length distribution, the overall wing's lift coefficient and the induced drag coefficient of the wing. From the Kutta-Joukowski Lift Theorem it is known that lift is directly proportional to circulation or vortex strength. Therefore, the theory must be capable of predicting the span-wise bound vortex strength per unit length distribution. The unknown vortex strength distribution, Γ (y), is approximated by a Fourier series as follows:

$$\Gamma(\theta)=2bV\sum_{n=1}^{N} A_n \sin(n\theta) \tag{1}$$

where:

$$y = -\frac{b}{2}\cos\theta \tag{2}$$

or

$$\theta = \cos^{-1}\left(-\frac{2y}{b}\right) \tag{3}$$

The basic problem is how to calculate the unknown Fourier series coefficients or amplitudes, A_n. The approximation using a Fourier series becomes more accurate as the number of terms, N, increases. However, for hand calculation we must limit the value of N to a very small number of 4 or less.
The lifting line equation that needs to be solved is

$$\frac{4b}{a(y).c(y)}\sum_{n=1}^{N} A_n \sin(n\theta(y)) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta(y))}{\sin(\theta(y))} = \alpha_G(y) - \alpha_0(y) \tag{4}$$

The above equation can be rewritten as follows:

$$\sum_{n=1}^{N}\left(\left(\frac{4b}{a(y).c(y)} + \frac{n}{\sin(\theta(y))}\right)\sin(n\theta(y))\right).A_n = \alpha_G(y) - \alpha_0(y) \tag{5}$$

Let us now define the following quantities

$$C(y,n)=\left(\frac{4b}{a(y).c(y)}+\frac{n}{\sin(\theta(y))}\right)\sin(n\theta(y)) \tag{6}$$

$$\mathrm{A(n)} = A_n \tag{7}$$

$$\mathrm{D(y)} = \alpha_G(y)-\alpha_0(y) \tag{8}$$

Equation (5) can now be written as

$$\sum_{n=1}^{N} C(y,n).A(n)=D(y) \tag{9}$$

The above equation contains N unknowns, namely A(n) for n=1 to N. It is therefore necessary to apply equation (9) at N different control points or values of distance along the span, y, so that we have a system of equations that can be solved simultaneously to calculate the values of A(n). The points chosen should not include the wing tips, since regardless of the values of the Fourier coefficients, the vortex strength distribution equation (1) is always satisfied at those points. Selecting those 2 points will not provide any new information regarding the values of the Fourier amplitudes. It is also recommended that the midpoint (y = 0) should also not be selected as a control point, for similar reasons. To get the most accurate result for a given number of control points, the following method for selecting the control points is recommended. Firstly, N should be chosen to be an even integer, such that N = 2M, say.

The N points along the span are chosen so that they are equally spaced. In other words the span is divided up into N equal intervals, and the midpoint of each interval is chosen to be a control point. The port (left) wing tip is located at y = -b/2 whereas the starboard (right) wing tip is located at y = b/2. The coordinates of the control points are then given as follows. For each value of k, from 1 to N, we have

$$y(k)=y_k=-\frac{b}{2}\left(1-\frac{2k-1}{N}\right) \tag{10}$$

Note that the control points are symmetrical about the plane of symmetry, y = 0. In other words we have y(k) = y(N+1-k). If the lift per unit span length distribution is symmetrical about y = 0, then its value at y(k) is equal to its value at y(N+1-k). Furthermore, it can be shown that the Fourier amplitudes with even indexes in equation (1) are all zero

$$A_{2m}=0 \qquad \text{for } m= 1, 2,\ldots,M \text{ (where } M = N/2)$$

Equation (1) can always be rewritten in a way, which separates the even terms from the odd terms as follows

$$\Gamma(\theta)=\sum_{n=1}^{N} A_n \sin(n\theta) = 2bV\left[\sum_{n=1}^{M} A_n \sin((2m-1)\theta)+\sum_{n=1}^{M} A_n \sin(2m\theta)\right] \tag{11}$$

Therefore, if the load distribution is symmetrical then equation (11) is simplified to

$$\Gamma(\theta)= 2bV\sum_{n=1}^{M} A_{2m-1} \sin((2m-1)\theta) = 2bV\sum_{n=1}^{M} A\left(2m-1\right)\sin((2m-1)\theta) \tag{12}$$

In the discussion below it is always assumed that the load distribution is symmetrical.

The Lifting Line Equation, which is the system of equations (9), that must be solved to calculate the Fourier coefficients, can now be written as follows

$$\sum_{m=1}^{M} C(k,2m-1).A(2m-1)=D(k) \qquad \text{for } k=1, 2, \ldots, M \tag{13}$$

where the values of y are given by equation (10) and

$$\theta_k=\theta\left(k\right)=\theta\left(y_k\right)=\cos^{-1}\left(1-\frac{2k-1}{2M}\right) \tag{14}$$

$$C\left(k,2m-1\right)=C(\theta_k,2m-1)=\left(\frac{4b}{a(k).c(k)}+\frac{2m-1}{\sin(\theta_k)}\right)\sin((2m-1)\theta_k) \tag{15}$$

$$D\left(k\right)=\alpha_G(k)-\alpha_0(k)=\alpha_G(y_k)-\alpha_0(y_k) \tag{16}$$

It should be noted that c(k) is the aerofoil's chord length at the station y(k), or $\theta\left(k\right)$, whereas $\alpha_G\left(k\right)$ and $\alpha_0\left(k\right)$ are the geometric and zero lift angle of attacks at y(k). The geometric angle of attack may vary as a function of y if the wing is given a geometric twist. A wing without twist is one where the geometric angle of attack is constant for all values of y, such that the leading edge and the trailing edge of the wing are straight lines, which lie on the same horizontal plane when $\alpha_G = 0$. A wing may be given a washout, where the wing is twisted such that the leading edge of the wing tip aerofoil is now lower than the leading edge of the root aerofoil (the root aerofoil is the aerofoil located at the plane of symmetry if it is imagined that the fuselage is not there and the two halves of the wing meet at the plane of symmetry). A wing with washin is one where the leading edge of the tip aerofoil is now higher than the leading edge of the root aerofoil, whereas the trailing edge of the wing remains on the horizontal plane. It follows, therefore, that the chord of the aerofoil at y may have negative or positive geometric angle of attack values when $\alpha_G = 0$ at the wing root, depending on whether the wing has a washout or a washin.

Let the height difference between the leading edge of the wing tip aerofoil from the leading edge of the root aerofoil is h_{tip}, which is negative for washout and positive for washin. It should be noted that the leading edge of the wing is required to remain as a straight line. Therefore, the twist angle or the geometric angle of attack at y relative to the geometric angle of attack at the wing root can be calculated as follows

$$\beta(k) = \sin^{-1}\left(\frac{h}{c}\right) = \sin^{-1}\left(-\frac{2y}{bc}.h_{tip}\right) = \sin^{-1}\left(\left(1 - \frac{2k-1}{N}\right)\frac{h_{tip}}{c}\right) \tag{17}$$

Let the angle of attack of the wing or the aircraft be denoted by the angle of attack at the wing root, and is given the symbol of α. This angle obviously can be varied and represents the attitude of the aircraft (when the aircraft is at a level cruising flight this angle may have a small positive value of not more than 3 degrees). Using this definition we can now calculate the geometric angle of attack at y as follows

$$\alpha_G(k) = \alpha + \beta(k) \tag{18}$$

Equation (16) can now be rewritten as follows

$$D(k) = \alpha - \alpha_0(k) + \beta(k) \tag{19}$$

A wing may be given an aerodynamic twist as well as a geometric twist. This means that the aerofoil shape at the wing root is different from that at the wing tip. The shape of the aerofoil in between the two limiting stations is then determined by insisting that the wing cross-section should have a smoothly varying shape along the span wise direction. Since the aerofoil shape at the wing tip is different from that at the root, therefore the value of the sectional lift coefficient as well as its zero lift angle of attack may also vary along the span wise direction. Provided the variation of $a(y)$ and $\alpha_0(y)$ are given, equation (15) can still be used to compute the matrix coefficients, $C(k, 2m-1)$, thus the lifting line theory can handle such a problem.

The theory can also handle the problem involving a variation in the chord length of the sections as a function of y, as long as the functional form of c(y) is given. This means that the theory is also applicable for analysing tapered wing shape, so long as the quarter chord line is normal or almost normal to the aircraft's longitudinal axis. Obviously the theory is not valid for a highly swept wing. For swept wings we should use vortex lattice method, but this will not be discussed here.

The taper ratio is the ratio of the chord length of the wing tip aerofoil to that of the root aerofoil. Normally the value is less than 1, i.e. the sectional chord length decreases with increasing distance away from the plane of symmetry, or with increasing magnitude of y. The taper ratio is given the symbol of λ and thus

$$\lambda = \frac{c_{tip}}{c_{root}} \tag{20}$$

The chord length at y is then given by the following equation

$$c(y) = c_{root} - 2\frac{c_{tip} - c_{root}}{b}.y$$

For computational purpose the above equation is more specifically written as follows

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b}y(k)\right) = \frac{c_{tip}}{\lambda}\left(1 - 2\frac{\lambda - 1}{b}y(k)\right) \tag{21}$$

The specification for aerodynamic twist is quite complicated, since it requires knowledge of the shape of the aerofoil section at each station y along the span. In the absence of such information, we can simplify the problem somewhat by requiring that the tip aerofoil differs only slightly from the root aerofoil such that the lift coefficient and zero angle of attack at y are given by the following linear relationships

$$a(k) = a_{root} - 2\frac{a_{tip} - a_{root}}{b}y(k) \tag{22}$$

$$\alpha_0(k) = \alpha_{0,root} - 2\frac{\alpha_{0,tip} - \alpha_{0,root}}{b}y(k) \tag{23}$$

Obviously the properties of the root and tip aerofoils must be supplied as inputs.
The procedure for the application of the lifting line theory in evaluating the aerodynamic performance of any kind of wing shape (with certain limitations) can now be summarized as follows.
The following data must be supplied:

1. The wing's angle of attack, α, and either the value of the wing span, b, the wing area, S, or the Aspect Ratio. These quantities are related as follows
$$b = \frac{S}{c_{av}} \quad \text{and} \quad AR = \frac{b^2}{S} = \frac{b}{c_{av}}$$
2. Any 2 of the following 3 quantities: root chord, tip chord and taper ratio or c_{root}, c_{tip} and λ. The relationship involving the 3 parameters is $\lambda = \frac{c_{tip}}{c_{root}}$.
   The average chord can be calculated as follows $c_{av} = \frac{1}{2}c_{root}(1+\lambda) = \frac{1}{2\lambda}c_{tip}(1+\lambda)$
3. The lift curve slope and zero lift angle of attack of the root aerofoil, a_{root} and $\alpha_{0,root}$, and also those of the tip aerofoil, a_{tip} and $\alpha_{0,tip}$.
4. The value of either the tip aerofoil leading edge height relative to that for the root aerofoil, h_{tip}, or the twist angle of the tip aerofoil relative to the root aerofoil, β_{tip}.

If the twist angle of the tip aerofoil relative to the root aerofoil is given, say β_{tip}, then the leading edge height difference should be calculated as follows

$$h_{tip} = c_{tip}.\sin\beta_{tip}$$

The computational procedure can then be described as follows

1. Select the value of the number of control points on the port wing M.
2. For k = 1, 2, 3, .., M calculate the following

$$y(k) = -\frac{b}{2}\left(1 - \frac{2k-1}{2M}\right) \tag{24}$$

$$\theta(k) = \cos^{-1}\left(-\frac{2y(k)}{b}\right) \tag{25}$$

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b}y(k)\right) \tag{26}$$

$$a(k) = a_{root} - 2\frac{a_{tip} - a_{root}}{b}y(k) \tag{27}$$

$$\alpha_0(k) = \alpha_{0,root} - 2\frac{\alpha_{0,tip} - \alpha_{0,root}}{b}y(k) \tag{28}$$

$$\beta(k) = \sin^{-1}\left(-\frac{2y(k)}{b.c(k)}.h_{tip}\right) \tag{29}$$

$$D(k) = \alpha - \alpha_0(k) + \beta(k) \tag{30}$$

For each value of m =1, 2, …, M calculate the matrix coefficients

$$C(k,m) = \left(\frac{4b}{a(k).c(k)} + \frac{2m-1}{\sin(\theta(k))}\right).\sin\left((2m-1).\theta(k)\right) \tag{31}$$

3. Now solve the following system of simultaneous equations

$$\sum_{m=1}^{M} C(k,m).A(m) = D(k) \qquad \text{for k = 1, 2, …, M} \tag{32}$$

A simple direct method for computing A(m), the solutions of (32), is the Gaussian Elimination Method. Other methods, such as the Jacobi or Gauss-Seidel iterative methods may also be used.

4. After the Fourier coefficients, A(m), have been calculated we can now compute the non-dimensional wing load distribution as follows

$$\Gamma_{ND}(y) = \frac{\Gamma(y)}{2bV} = \sum_{1}^{M} A(m).\sin\left((2m-1).\theta\right)$$
$$\theta(y) = \cos^{-1}\left(-\frac{2y}{b}\right) \tag{33}$$

5. The wing's lift coefficient can be calculated as follows

$$C_L = \pi.AR.A(1) \tag{34}$$

6. The Oswald efficiency factor, e, is

$$e = \frac{1}{1+\delta} \quad \text{where} \quad \delta = \sum_{m=2}^{M}(2m-1).\left(\frac{A(m)}{A(1)}\right)^2 \tag{35}$$

7. Induced drag coefficient C_{Di} is

$$C_{Di} = \frac{C_L^2}{\pi.AR.e}$$

The formula for Induced drag can also be written as

$C_{Di} = k.C_L^2$ where k is the induced drag factor and $k = \dfrac{1}{\pi.AR.e}$

<u>Worked Examples</u>

The Problem:

A hypothetical, conventional small aircraft has a wing area of 50 square meter and an aspect ratio of 8. The wing has no dihedral, is unswept at its quarter chord line and is not twisted aerodynamically, or its cross section is the same aerofoil shape all along the span. The tapered wing with a taper ratio of 0.6 is twisted geometrically. The geometric twist is such that the tip aerofoil section is at an incidence of –2.9 degrees when the root section is at zero degree incidence. The aerofoil's zero lift angle of incidence is –2.0 degrees and the aerofoil's lift coefficient curve has a slope of 6 per radian. If the wing's angle of incidence is 2 degrees while the aircraft is cruising, calculate the following

(i) Calculate the wing's span and mean chord lengths. Also calculate the root chord and the tip chord lengths.

(ii) At the span wise stations y = ± 7.071 m and y = ± 3.827 m calculate the local chord lengths of the wing (note that the aircraft's plane of symmetry is located at the span's mid point or at the station y = 0)

(iii) Calculate the geometric angle between the local chord line at each of the span wise station stated above and the chord line at the root section, due to the twist.

(iv) Calculate the overall lift coefficient of the aircraft at cruise condition using the lifting line theory. Hint : Use 2 of the y-stations given above, say y = 7.071 m and 3.827 m, or y = -7.071 m and y = -3.827 m as your control points in the calculation. Would you get different answers if you were to do the calculation using all 4 control points along the whole span, as compared to only using 2 points along the semi span and taking advantage of the wing loading symmetry? Explain.
Also calculate the Oswald factor and the induced drag coefficient.

The Answer:

(1). Wing span, b, and mean chord length, c_{av}.

$$b = \sqrt{S.AR} = \sqrt{50x8} = 20\,m$$

$$c_{av} = \sqrt{\frac{S}{AR}} = \sqrt{\frac{50}{8}}m = 2.5\,m$$

Root chord and tip chord

$$c_{root} = \frac{2}{1+\lambda}.c_{av} = \frac{2}{1+0.6}x2.5 = 3.125\,m$$

$$c_{tip} = \lambda.c_{root} = 0.6x3.125 = 1.875\,m$$

(2). Local chord lengths at the following stations
y(1) =-7.071 m, y(2) = -3.827 m, y(3) = 3.827 m and y(4) = 7.071

Port wing (y negative): $c(y) = c_{root} + 2\frac{c_{root} - c_{tip}}{b}.y$, for –b/2 <y< 0

$$c(1) = 3.125 + 0.125y = 2.2411\,m$$

$$c(2) = 3.125 - 0.4784 = 2.6466\,m$$

Starboard wing (y positive): $c(y) = c_{root} - 2\frac{c_{root} - c_{tip}}{b}.y$, for 0 <y< b/2

$$c(3) = 3.125 - 0.4784 = 2.6466\,m = c(2)$$

$$c(4) = 3.125 - 0.8839 = 2.2411\,m = c(1)$$

(3). Twist angle distribution

Twist angle at the wing tip: $\beta_{tip} = -2.9^0$ thus

$h_{tip} = c_{tip}.\sin\beta_{tip} = 1.875x\sin\left(-2.9^0\right) = -0.09486$

Port wing (y negative): $\beta(y) = \sin^{-1}\left(-\frac{2h_{tip}}{b}.\frac{y}{c(y)}\right) = \sin^{-1}\left(0.009486\frac{y}{c(y)}\right)$

$$\beta(1) = \sin^{-1}\left(-0.009486x\frac{7.071}{2.2411}\right) = \sin^{-1}(-0.02993) = -1.72^0$$

$$\beta(2) = \sin^{-1}\left(-0.009486x\frac{3.827}{2.6466}\right) = \sin^{-1}(-0.01372) = -0.79^0$$

Due to symmetry we have

$$\beta(3) = \beta(2) = -0.79^0 \text{ and } \beta(4) = \beta(1) = -1.72^0$$

(4). Lifting Line theory

Port wing: $\theta = \cos^{-1}\left(\frac{2|y|}{b}\right)$ and starboard wing: $\theta = \cos^{-1}\left(-\frac{2|y|}{b}\right)$

$$\theta(1) = \cos^{-1}\left(\frac{2x7.071}{20}\right) = \cos^{-1}(0.7071) = 45^0$$

$$\theta(2) = \cos^{-1}\left(\frac{2x3.827}{20}\right) = \cos^{-1}(0.3827) = 67.5^0$$

$$\theta(3) = \cos^{-1}\left(-\frac{2x3.827}{20}\right) = \cos^{-1}(-0.3827) = 112.5^0$$

$$\theta(4) = \cos^{-1}\left(-\frac{2x7.071}{20}\right) = \cos^{-1}(-0.7071) = 135^0$$

Lift curve slope is constant or $a(k) = 6\,rad^{-1}$ for k = 1, 2, 3 and 4.

Similarly, the zero angle of attack is $\alpha_0(k) = -2^0$ for k = 1, 2, 3 and 4

The wing's angle of incidence is $\alpha = 2^0$

The matrix coefficients are given by the following equation

$$C(k,m) = \left(\frac{4b}{a.c(k)} + \frac{2m-1}{\sin\theta(k)}\right).\sin\left((2m-1)\theta(k)\right)$$

$$C(k,m)=\left(\frac{13.333}{c(k)}+\frac{2m-1}{\sin\theta(k)}\right).\sin\left((2m-1)\theta(k)\right)$$

$$D(k)=\alpha-\alpha_0+\beta(k)=2^0-\left(-2^0\right)+\beta(k)=4^0+\beta(k)$$

Taking advantage of the symmetry of the wing loading distribution we have M = 2 and we can use m =1 and 2 (port wing only) or m =3 and 4 (starboard wing only). If port wing control points only are chosen, then we have

$$C(1,1)=\left(\frac{13.333}{c(1)}+\frac{1}{\sin(\theta(1))}\right).\sin(\theta(1))=\left(\frac{13.333}{2.2411}+\frac{1}{0.7071}\right)x0.7071=5.2069$$

$$C(2,1)=\left(\frac{13.333}{c(2)}+\frac{1}{\sin(\theta(2))}\right).\sin(\theta(2))=\left(\frac{13.333}{2.6466}+\frac{1}{0.9239}\right)x0.9239=5.6544$$

$$C(1,2)=\left(\frac{13.333}{c(1)}+\frac{3}{\sin(\theta(1))}\right).\sin(3x\theta(1))=\left(\frac{13.333}{2.2411}+\frac{3}{0.7071}\right)x0.7071=7.2069$$

$$C(2,2)=\left(\frac{13.333}{c(2)}+\frac{3}{\sin(\theta(2))}\right).\sin(3x\theta(2))=\left(\frac{13.333}{2.6466}+\frac{3}{0.9239}\right)x-0.3827=-3.1706$$

$$D(1)=4+\beta(1)=2.28^0=0.039794\,rad$$

$$D(2)=4+\beta(2)=3.21^0=0.056025rad$$

The system of equations is as follows

$$C(1,1).A(1)+C(1,2).A(2)=D(1)$$
$$C(2,1).A(1)+C(2,2).A(2)=D(2)$$

This can be solved using Gaussian Elimination method (see Appendix) as follows

1.The following procedure should be carried out for each value of k, starting from k=1, then k=2,3,…up to k=M-1 after each loop is completed

1.1 Normalized the k^{th} row .

For each value of m, starting from m=k+1 to M, i.e. for m=k+1,k+2,..,M

$$C'(k,m)=\frac{C(k,m)}{C(k,k)}$$

Then

$$D'(k) = \frac{D(k)}{C(k,k)}$$

The result is

$$A(1) + C'(1,2).A(2) = D'(1)$$

$$C(2,1).A(1) + C(2,2).A(2) = D(2)$$

where $C'(1,2) = \dfrac{C(1,2)}{C(1,1)}$ and $D'(1) = \dfrac{D(1)}{C(1,1)}$

1.2 Eliminate all elements in the k^{th} column of all rows under the k^{th} row

For each value of j, starting with j=k+1 then j=k+2 etc up to j=M

$$C'(j,m) = C(j,m) - C(j,k).C'(k,m)$$

Then

$$D'(j) = D(j) - C(j,k).D'(k)$$

The result is

$$A(1) + C'(1,2).A(2) = D'(1)$$

$$C'(2,2).A(2) = D'(2)$$

where $C'(2,2) = C(2,2) - C(2,1).C'(1,2)$ and $D'(2) = D(2) - C(2,1).D'(1)$

Even though in the above example the matrix is only 2X2 or M=2, the procedure is quite general and can be applied for any value of M.

The procedure described above is the elimination part of the Gauss method. Note that the end result is a “triangular matrix” with all diagonal elements having the value of unity, and all elements below the diagonal are zeros.

Now we shall describe the back substitution part of the method.

After the last elimination process has been done, we will have the last equation to be of the form

$$C^*(M,M).A(M) = D^*(M)$$

This can immediately be solved for A(M) as follows

$$A(M) = D^*(M)/C^*(M,M)$$

Knowing the value of A(M), we can now use the equation directly above the M^{th} equation, which has the following general form

$$A(M-1) + C(M,M).A(M) = D(M-1)$$

The above equation can be solved immediately for A(M-1) as follows

$$A(M-1) = D^*(M-1) - C^*(M-1,M).A(M)$$

This process can be continued until all A(m) for m=M, M-1, M-2,…,1 have been computed. For A(M-2) the equation is as follows

$$A(M-2) = D^*(M-2) - C^*(M-2,M).A(M) - C^*(M-2,M-1).A(M-1)$$

The generalized formula for back substitution is as follows

First calculate $A(M)$:

$$A(M) = D^*(M)/C^*(M,M) \text{ and then}$$

For k = 1,2,…,M-1

$$A(M-k) = D^*(M-k) - \sum_{j=0}^{k-1} C^*(M-k, M-j).A(M-j)$$

For the example given we have the following calculated data

C(1,1) = 5.2069	C(1,2) = 7.2069	D(1) = 0.039794
C(2,1) = 5.6544	C(2,2) = -3.1706	D(2) = 0.056025

The matrix can be written as

$$\begin{bmatrix} 5.2069 & 7.2069 \\ 5.6544 & -3.1706 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \end{bmatrix} = \begin{bmatrix} 0.039794 \\ 0.056025 \end{bmatrix}$$

Elimination process gives the following result

Normalization

$$\begin{bmatrix} 1 & 1.3841 \\ 5.6544 & -3.1706 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \end{bmatrix} = \begin{bmatrix} 0.007643 \\ 0.056025 \end{bmatrix}$$

Elimination

$$\begin{bmatrix} 1 & 1.3841 \\ 0 & -10.9969 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \end{bmatrix} = \begin{bmatrix} 0.007643 \\ 0.012808 \end{bmatrix}$$

Then $A(2) = \dfrac{0.012808}{-10.9969} = -0.001165$ and

Back substitution

$$A(1) = 0.007643 - 1.3841x(-0.001165) = 0.009255$$

Therefore, the wing load distribution is given by the following

$$c.C_l = \frac{2\Gamma}{V} = 4b\left(A(1)\sin\theta + A(2)\sin 3\theta\right)$$

Thus $c.C_l = \dfrac{2\Gamma}{V} = 4b\left(0.00925\sin\theta - 0.001165\sin 3\theta\right)$

The total wing lift coefficient is

$$C_L = \pi.AR.A(1) = 8\pi x 0.00925 = 0.2325$$

The Oswald efficiency planform factor is

$$e = \frac{1}{1+3x\left(\frac{0.001165}{0.009255}\right)^2} = \frac{1}{1.04754} = 0.9546$$

The induced drag coefficient is

$$C_{Di} = \frac{C_L^2}{\pi.AR.e} = 0.04168C_L^2 = 0.002253$$

Let us now investigate the effect of the number of control points on accuracy of the result
Let a third point be chosen on the port wing such that $\theta(3) = 22.5^0$. The value of y(3) is then $y(3) = -\frac{b}{2}\cos\theta(3) = -9.2388$.

The chord length at the third station is $c(3) = 3.125 - 0125x9.2388 = 1.970$

The twist angle is $\beta(3) = \sin^{-1}\left(-0.009486x\frac{9.2388}{1.970}\right) = -2.55^0$

Therefore $D(3) = 4^0 - 2.55^0 = 1.45^0 = 0.025307\ rad$

The additional matrix coefficients required are given by the following equation

$$C(k,m) = \left(\frac{13.333}{c(k)} + \frac{2m-1}{\sin\theta(k)}\right).\sin\left((2m-1)\theta(k)\right)$$

$$C(1,3) = \left(\frac{13.333}{2.2411} + \frac{5}{0.7071}\right)x(-0.7071) = -9.2069$$

$$C(2,3) = \left(\frac{13.333}{2.6466} + \frac{5}{0.92388}\right)x(-0.3827) = -3.9992$$

$$C(3,1) = \left(\frac{13.333}{1.9702} + \frac{1}{0.38268}\right)x038268 = 3.5898$$

$$C(3,2) = \left(\frac{13.333}{1.9702} + \frac{3}{0.38268}\right)x0.92388 = 13.4950$$

$$C(3,3) = \left(\frac{13.333}{1.9702} + \frac{5}{0.38268}\right)x0.92388 = 18.3234$$

The system of equations for the 3 control points case is as follows

$$\begin{bmatrix} 5.2068 & 7.2069 & -9.2069 \\ 5.6544 & -3.1706 & -3.9992 \\ 3.5898 & 13.4950 & 18.3234 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \\ A(3) \end{bmatrix} = \begin{bmatrix} 0.039794 \\ 0.056025 \\ 0.025307 \end{bmatrix}$$

The elimination process gives the following results

$$\begin{bmatrix} 1 & 1.3841 & -1.7682 \\ 0 & -10.9969 & 5.99897 \\ 0 & 8.5264 & 24.67092 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \\ A(3) \end{bmatrix} = \begin{bmatrix} 0.0076427 \\ 0.012808 \\ -0.00213 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.3841 & -1.76821 \\ 0 & 1 & -0.54551 \\ 0 & 0 & 29.3221 \end{bmatrix} . \begin{bmatrix} A(1) \\ A(2) \\ A(3) \end{bmatrix} = \begin{bmatrix} 0.0076427 \\ -0.001165 \\ 0.007803 \end{bmatrix}$$

(details of how the calculations are carried out is given in the appendix)

Now the back substitution part

$$A(3) = \frac{0.007803}{29.3221} = 0.0002661$$
$$A(2) = -0.001165 - (-0.54551)x0.0002661 = -0.00102$$
$$A(1) = 0.007643 - (-1.76821)x0.0002661 - 1.3841x(-0.000102) = 0.009525$$

Wing's lift coefficient $C_L = \pi.AR.A(1) = 8\pi x 0.009525 = 0.2394$

Oswald efficiency factor

$$e = \frac{1}{1 + 3x\left(\frac{0.00102}{0.009525}\right)^2 + 5x\left(\frac{0.0002661}{0.009525}\right)^2} = \frac{1}{1 + 0.034403 + 0.039024} = 0.9631$$

Induced drag coefficient: $C_{Di} = \frac{C_L^2}{\pi.AR.e} = 0.04133 C_L^2 = 0.002368$

It can be observed that the values of lift coefficient and Oswald efficiency factor computed using Fourier series with 2 terms, differ from those obtained using Fourier series with 3 terms, by approximately 3 percentage points. However, the error of the induced drag is aproximately 5 percentage points. Obviously, if greater accuracy is required, then we have to use a Fourier series with more terms. From the results obtained in this example, it appears that the Fourier series approximation is only accurate to, let us say I percent, if the number of the Fourier coefficients is greater than 3.

Appendix

Solution of a system of simulltaneous linear equations using Gaussian Elimination method

The system of equations to be solved is as follows

5.2068 A(1) + 7.2069 A(2) – 9.2069 A(3) = 0.039794
5.6544 A(1) – 3.1706 A(2) – 3.9992 A(3) = 0.056025
3.5898 A(1) + 13.495 A(2) + 18.323 A(3) = 0.025307

The above system of equations can be written in matrix form as follows

$$\begin{vmatrix} 5.2068 & 7.2069 & -9.2069 & 0.039794 \\ 5.6544 & -3.1706 & -3.9992 & 0.056025 \\ 3.5898 & 13.495 & 18.323 & 0.025307 \end{vmatrix}$$

Normalization of first row

C(1,2) = 7.2069/5.2068 = 1.38413
C(1,2) = -9.2069 /5.2068 = -1.76821
D(1) = 7.2069/5.2068 = 0.0076427

The resulting matrix is

$$\begin{vmatrix} 1 & 1.3841 & -1.7682 & 0.0076427 \\ 5.6544 & -3.1706 & -3.9992 & 0.056025 \\ 3.5898 & 13.495 & 18.323 & 0.025307 \end{vmatrix}$$

Elimination for second and third rows

C(2,2) = - 3.1706 – 5.6544 x1.38413 = -10.9970
C(2,3) = - 3.9992 – 5.6544 x (-1.7682) = 5.99897
D(2) = 0.056025– 5.6544 x 0.0076427 = 0.01281

C(3,2) = 13.495 – 3.5898 x 1.38413 = 8.52625
C(3,3) = 18.3234 – 3.5898 x (-1.76821) = 24.67092
D(3) = 0.025307–3.5898 x 0.0076427 = -0.0021288

The resulting matrix is

$$\begin{vmatrix} 1 & 1.38413 & -1.7682 & 0.0076427 \\ 0 & -10.9970 & 5.99897 & 0.01281 \\ 0 & 8.52625 & 24.67092 & -0.0021288 \end{vmatrix}$$

Normalization for second row

C(2,3) = 5.99897/(-10.9970) = -0.54551
D(2) = 0.01281/(-10.9970) = -0.001165

The resulting matrix is

$$\begin{vmatrix} 1 & 1.3841 & -1.76821 & 0.0076427 \\ 0 & 1 & -0.54551 & -0.001165 \\ 0 & 8.52625 & 24.67092 & -0.0021288 \end{vmatrix}$$

Elimination for third row

C(3,3) = 24.67092 - 8.52625 x (-0.54551) = 29.3221
D(3) = -0.0021288 - 8.52625 x (-0.001165) = 0.007804

The resulting matrix is

$$\begin{vmatrix} 1 & 1.3841 & -1.76821 & 0.0076427 \\ 0 & 1 & -0.54551 & -0.001165 \\ 0 & 0 & 29.3221 & 0.007804 \end{vmatrix}$$

The above matrix represents the following system of equations

$$\begin{aligned} A(1) + 1.3841A(2) - 1.76821A(3) &= 0.0076427 \\ A(2) - 0.54551A(3) &= -0.001165 \\ 29.3221A(3) &= 0.007804 \end{aligned}$$

Now the values of the unknowns A(1), A(2) and A(3) can be calculated as follows

Back substitution

A(3) = 0.007804 / 29.3221 = 0.0002661

A(2) = -0.001165- (-0.54551) x 0.0002661 = -0.00102

A(1) = 0.0076427 – (-1.76821) x 0.0002661 - 1.38413 x (-0.00102) = 0.009525

Please report any error(s) to the author at hadi.winarto@rmit.edu.au
Last updated 27 May 2004